

Using Laplace Transforms to Build a Linear/Non-linear Speedometer

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ABSTRACT

Consider the design of a speedometer (as used in a car, for example) using conventional mechanical linkages and a rotating magnet. We begin by modeling a linear speedometer. Later we introduce real-life nonlinearities. With the system linear, Laplace transforms make the solution of the problem easy and complete compared to solutions using differential equations. In the second case, we analyze several nonlinear effects. Although Laplace transforms are only accurate for linear systems, they can be used as is for several real-world nonlinearities, or they can be adapted for use with any generic nonlinearity by using a lookup table. Analysis and computations are done using MATLAB.

INTRODUCTION

A conventional speedometer measures the speed of a vehicle using a rotating magnet [1]. The magnet is linked to one of the vehicle's axles and rotates as the axle rotates. Surrounding the magnet is a metal non-ferrous can. The can is free to rotate about a pivot point at its center. According to Faraday's law and the laws of magnetic force, the rotational speed of the magnet causes a constant torque to be applied to the metal can [2]. The faster the magnet spins, the stronger the magnetic torque. In a steady-state, the can has a displacement angle θ , and θ is set to equal zero when the speed is zero. As the speed increases, the magnetic torque increases and θ increases, until the magnetic torque is equal to the spring restoring force.

Design & Analysis

The construction of the speedometer as a system is shown in Figure 1. The speed of the vehicle (v) is converted to an angular axle speed ω . This is represented by an amplifier G_1 . This conversion is strictly linear and is given by $v = \omega * R$, and r is the radius of the tire (wheel). The radius of the tire varies greatly from vehicle to vehicle. We can assume $R = 0.45$ m for our purposes here. The degree of "roundness" of the tire also plays a role here. If the tire is not fully inflated, there is a small correction factor required: $v = \omega * \alpha * r$, and $\alpha \leq 1$. For our purposes, we can ignore alpha.

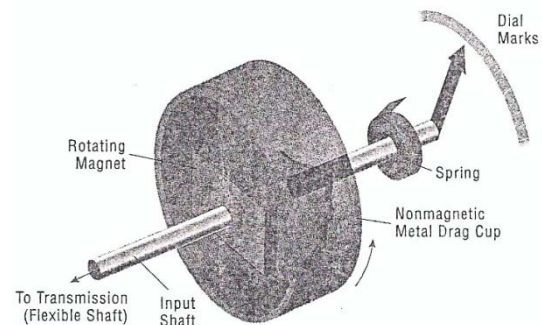


Figure 1: Construction of a Speedometer [1]

$$V \xrightarrow{\frac{1}{R}} W \xrightarrow{\text{Time Lag}} W_R \longrightarrow T \xrightarrow{G} \Theta$$

The turning axle is coupled by mechanical linkage (gears usually) to a permanent magnet which rotates. The rotational speed of the magnet is ω_R . If the linkage is perfect, then $\omega = \omega_R$. In real-life, there is usually a time delay (τ) involved.

$$W_R|_{s=0} = W|_{s \rightarrow 0}$$

$$W_R = W \cdot \frac{1}{1+s\tau} \text{ where } \tau = 2.4 \text{ m sec}$$

$$\text{Let } = \frac{1}{s}, \text{ then we have } W_R = \frac{1}{s} \cdot \frac{1}{1+s\tau} = \frac{1}{s} - \frac{1}{s+1/\tau}$$

$$\text{Converting to the time domain, we have } \omega_R(t) = 1 - e^{-t/\tau}$$

$$\text{We also know that the torque } \tau = J\ddot{\theta} + b\dot{\theta} + k\theta$$

$$\text{In s-domain we get } T = (Js^2 + bs + k)\Theta$$

$$G = \frac{\Theta}{T} = \frac{1}{Js^2 + bs + k}$$

The above equation, is in a general form that can be adapted to varying inputs and changing coefficients.

$$\text{Where } J = mR^2 = (0.002)(0.45)^2 = 4.05 * 10^{-4} \text{ and } k = \frac{\Delta F}{\Delta \theta} = \frac{0.002 * 9.81}{\pi} = 6.25 * 10^{-3}$$

$$\Theta = \frac{T}{4.05 * 10^{-4} * s^2 + bs + 6.25 * 10^{-3}}$$

Solving the equation gives us an over-damped, under-damped, or critically damped response for the output relative to the input. At this point, we make an assumption based on the workings of a real speedometer: we assume the response is critically damped. In other words, when we change speed in our car, the dial on the speedometer changes to the new setting without overshooting the next speed marking.

Let us assume that T is a unit step function. In order of the two poles P_1, P_2 to be real

$$6.1 * 10^6 * b^2 - 61.72 > 0 \Rightarrow b > 3.2 * 10^{-3}$$

The time taken for the speedometer needle to rise from 0 to 20 mph is approximately 2.5secs

$$\therefore 5\tau = 2.5 \Rightarrow \tau = 0.5 \text{ We have the poles } P_1, P_2$$

$$P_1, = \frac{-2.47 * 10^3 - \sqrt{6.1 * 10^6 * b^2 - 61.72}}{2} \text{ and } P_2, = \frac{-2.47 * 10^3 + \sqrt{6.1 * 10^6 * b^2 - 61.72}}{2}$$

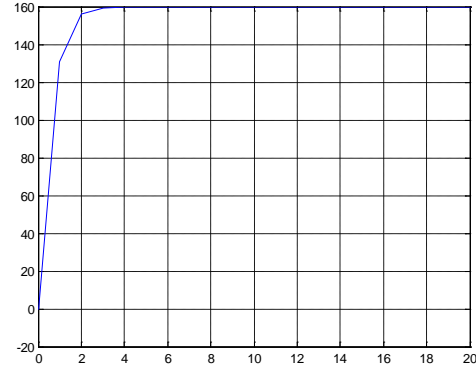


Figure 2: Step response to the system

$\therefore |P_1| < |P_2|$ Hence P_1 has the larger time constant

$$\tau = \frac{1}{P_1} = 0.5 \Rightarrow b = 3.93 * 10^{-3}$$

$$\therefore \Theta = \frac{1}{s(4.05 * 10^{-4} * s^2 + 3.93 * 10^{-3} * s + 6.25 * 10^{-3})} \text{ where } T = \frac{1}{s}$$

Note: using a step change in velocity is non-realistic. However, it keeps the total function of the speedometer linear. We will explore the more realistic nonlinear case later.

Converting to time domain we get $\theta = 160 - 216.3 * e^{-2t} + 56.3 * e^{-7.7t}$

Figure 1 shows the how the displacement of the needle θ varies when the torque T is applied as a step function. Hence it is observed that θ quickly climbs to a maximum value and settles there. Hence it can be said that θ tracks the torque T .

Case 1A: Torque linearly proportional to Speed – In the simplest case the rotation of the magnet is linearly proportional to the torque.

$$V \xrightarrow{\frac{1}{R}} W \xrightarrow{\text{Time Lag}} W_R \xrightarrow{\bar{k}} T \xrightarrow{G} \Theta$$

$$W_r \sim T \Rightarrow \frac{T}{W_r} = \bar{k}$$

$$\frac{\Theta}{V} = \frac{\bar{k} C}{R} \frac{1}{(1 + s\tau)} \frac{1}{(Js^2 + bs + k)}$$

At speed $v(\text{min}) = 0$, $\theta = 0$ and at $v(\text{max}) = 120$, $\theta = \pi$

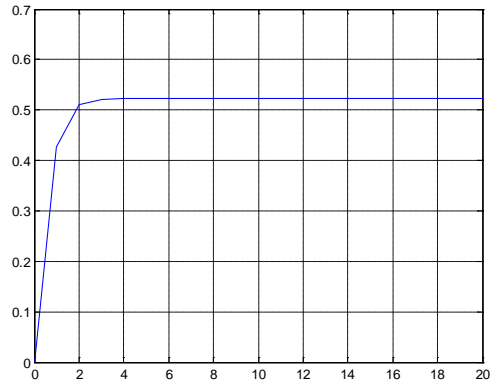


Figure 3: Response to a linear increase in speed.

At $v = 120$; $V = \frac{120}{s}$ and $\theta|_{t=0} = \pi = s\Theta|_{s=0}$

$$\pi = s \cdot \frac{120}{s} \left[\frac{\bar{k} C}{R} \frac{1}{(1 + s\tau)} \frac{1}{(Js^2 + bs + k)} \right]_{s=0} \Rightarrow \bar{k} C = 2.34\pi * 10^{-5}$$

$$\therefore \frac{\Theta}{V} = \left(\frac{2.34\pi * 10^{-5}}{0.45} \right) \left(\frac{1}{1 + 2.4 * 10^{-3} * s} \right) \left(\frac{1}{4.05 * 10^{-4} * s^2 + 3.93 * 10^{-3} * s + 6.25 * 10^{-3}} \right)$$

To test the above equation we can check that when $V = 20$ mph; it is expected that $\theta = \pi/6$ and $V = \frac{20}{s}$.

In the graph shown in Figure 3, we have assumed that the Torque on the magnet was linearly proportional to the speed. Hence when a speed of 20 mph is applied it is expected that the angle of displacement of the speedometer needle $\theta = \pi/6$ which can be noted from the graph. Hence the equation developed is accurate, at least in the linear case.

$$\text{Steady state error } (e_{ss}) = r_{\infty} - y_{\infty}$$

$$(e_{ss}) = r_{\infty} - y_{\infty} = \pi/6 - 0.523 \Rightarrow 0.11\%$$

The equation produces very accurate results with a steady state error of less than 0.1%.

Case 1B: Exponential increase in Speed. Unlike in the previous case where the input speed was modeled as a ramp input, but in reality the speed increases exponentially and hence can be represented by the equation.

$$v = v_0 (1 - e^{-t/x})$$

If the final speed is say $V = 20$ mph, then it takes 45 seconds for the needle to climb from 0 to 20 mph. Then $x = 0.9$

$$\therefore v = 20 (1 - e^{-t/0.9}) \text{ and in the s - domain we have } V = \frac{20}{s} - \frac{20}{s+t/0.9}$$

$$\Theta = \left(\frac{2.34\pi * 10^{-5}}{0.45} \right) \left(\frac{1}{1 + 2.4 * 10^{-3} * s} \right) \left(\frac{1}{4.05 * 10^{-4} * s^2 + 3.93 * 10^{-3} * s + 6.25 * 10^{-3}} \right) \left(\frac{20}{s} - \frac{20}{s + t/0.9} \right)$$

$$\text{Steady state error } (e_{ss}) = r_{\infty} - y_{\infty}$$

$$(e_{ss}) = r_{\infty} - y_{\infty} = \pi/6 - 0.523 \Rightarrow 0.11\%$$

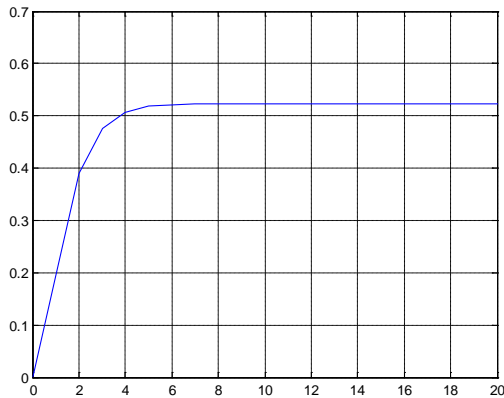


Figure 4: Response to an exponential increase in Speed

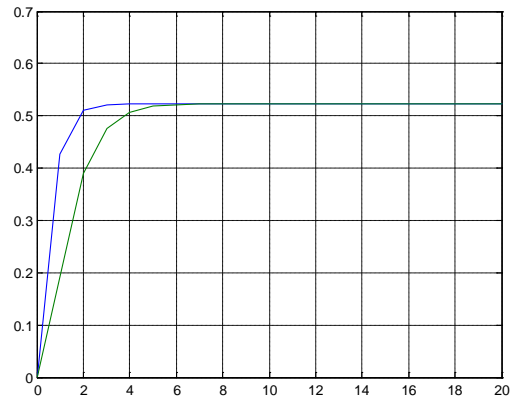


Figure 5: Comparison between case 1A and 1B

Figure 4 graph shows a more realistic increase in Speed (v) to 20 mph. accordingly, we see that the increase in displacement of the needle θ more gradual as it levels of at $\pi/6$, when compared to the previous case.

The figure 5 plot shows by how much the gradual increase in v lags when compared to the first case. The rate of change of θ is much less in curve-2 than in curve-1. As $t \rightarrow \infty$ they both settle at $\pi/6$ as V also settles at 20 mph.

Case 1C: Jerking Forward – Jerking forward is defined in its technical sense as the change in acceleration with time. Hence acceleration (a) \neq constant.

Let $v = a_0 t$; we also know that $a = \frac{dv}{dt}$

$$\therefore \frac{dv}{dt} = a_0 t \Rightarrow v = \frac{1}{2} a_0 t^2 + a$$

It is known that $v|_{t=0} = 0 \Rightarrow a = 0$. Hence we will derive the necessary jerk motion equation is now.

$$v = At^2$$

The function is clipped at $t = 3$ sec $\therefore 5\tau = 3 \Rightarrow \tau = \frac{3}{5} = 0.6$

$$V = (-20 + At^2)e^{-t/\tau} + V_\infty$$

Now we have $V_\infty = 20$ and $V_{t=3} = 0.9 * 20 = 18$. Substituting in the above equation we get $A = -30.76$

$$\therefore V = (-20 - 30.76t^2)e^{-t/\tau} + V_\infty$$

The equation for the 'jerking' motion is $v(t) = 20 - (30.76t^2 + 20)e^{-t/0.6}$

Converting to s-domain, we have $V = \frac{20}{s} - \frac{20}{s+\frac{5}{3}} - \frac{61.52}{(s+\frac{5}{3})^3}$

We know that;

$$\frac{\Theta}{V} = \frac{\bar{k} C}{R} \frac{1}{(1 + s\tau)} \frac{1}{(Js^2 + bs + k)}$$

$$\Theta = \left(\frac{2.34\pi * 10^{-5}}{0.45} \right) \left(\frac{1}{1 + 2.4 * 10^{-3} * s} \right) \left(\frac{1}{4.05 * 10^{-4} * s^2 + 3.93 * 10^{-3} * s + 6.25 * 10^{-3}} \right) \left(\frac{20}{s} - \frac{20}{s + 1.67} - \frac{61.52}{(s + 1.67)^3} \right)$$

$$\therefore \Theta = 0.523 + 117.2e^{-2t} - 0.072e^{-7.7t} + 1.9 * 10^{-7}e^{-416.7t} - (6.12t^2 + 38.3t - 117.7)e^{-1.67t}$$

Steady state error (e_{ss}) = $r_\infty - y_\infty$

$$(e_{ss}) = r_\infty - y_\infty = \pi/6 - 0.523 = 0.11\%$$

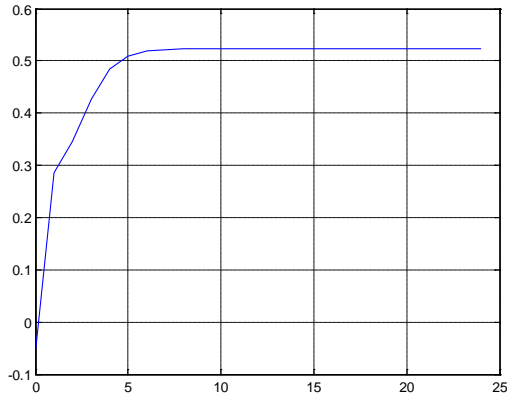


Figure 6: Response to 'Jerking' Speed input

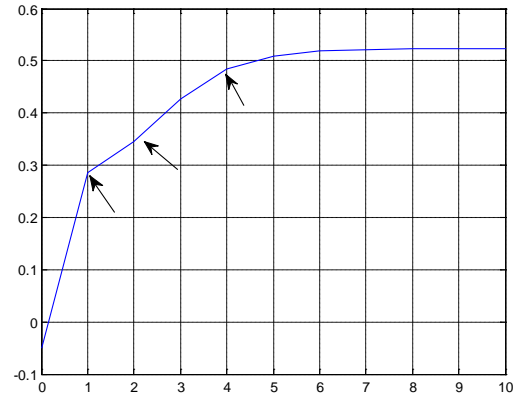


Figure 7: Marked changes in $d\theta/dt$

As $t \rightarrow \infty$ it is observed in Figure 6 that $\theta \rightarrow \pi/6$ as was expected.

In figure 7, when t is small, V is based on the 'jerking' motion. From the graph it can be observed that dv/dt changes thrice before settling at $\pi/6$. The jerks, changes in acceleration can be clearly seen in the graph.

Case 2: Quadratic Torque. In this case we assume that the torque is quadratic in relation to the speed.

Let the quadratic torque be 'n' and $\mathcal{L}(n) = T$

Assume $n = w_R k_1 + k_2 w_r^2$ where $k_1 = 8$ and $k_2 = 0.2$

$$T = k_1 W_R + k_2 \mathcal{L} W_R^2$$

$$\text{And } \frac{W_R}{V} = \frac{1}{R} \left(\frac{1}{1+s\tau} \right)$$

We have $v = v_0 (1 - e^{-t/0.6})$; at $t = 0, v = 0$ and $t = \infty, v = v_0 = 20 \text{ mph}$

$$W_R = \frac{20}{0.45} \left(\frac{1}{1 + s * 2.4 * 10^{-3}} \right) \left(\frac{1}{s} - \frac{1}{s + 1.67} \right) \Rightarrow w_R(t) = 44.4 + 0.178e^{-416.7t} - 44.6e^{-1.67t}$$

$$\therefore n/k_1 = (44.4 + 0.178e^{-416.7t} - 44.6e^{-1.67t}) + \frac{1}{40} (44.4 + 0.178e^{-416.7t} - 44.6e^{-1.67t})^2$$

$$\Rightarrow \frac{T}{k_1} = \frac{93.7}{s} + \frac{0.573}{s + 416.7} - \frac{143.6}{s + 1.33} - \frac{0.2}{s + 418.4} + \frac{7.9 * 10^{-4}}{s + 833.4} + \frac{49.73}{s + 3.33}$$

$$\Theta = TGk^* = 8 \left(\frac{93.7}{s} + \frac{0.573}{s + 416.7} - \frac{143.6}{s + 1.33} - \frac{0.2}{s + 418.4} + \frac{7.9 * 10^{-4}}{s + 833.4} + \frac{49.73}{s + 3.33} \right) \left(\frac{1}{4.05 * 10^{-4} * s^2 + 3.93 * 10^{-3} * s + 6.25 * 10^{-3}} \right) k^*$$

Using the final value theorem $s\Theta|_{s \rightarrow 0} = \Theta|_{t \rightarrow \infty}$

$$\frac{\pi}{6} = 8 * 93.7 \left(\frac{1}{6.25 * 10^{-3}} \right) k^* \Rightarrow k^* = 1.39\pi * 10^{-6}$$

$$\begin{aligned} \therefore \Theta = & 0.523 + 6.29e^{-2t} - 3.7 * 10^{-3}e^{-7.7t} + 1.9 \\ & * 10^{-7}e^{-416.7t} + 9.9 * 10^{-11}e^{-833.4t} \\ & - 60.7e^{-1.67t} - 74e^{-3.33t} \end{aligned}$$

Steady state error (e_{ss}) = $r_{\infty} - y_{\infty}$

$$(e_{ss}) = r_{\infty} - y_{\infty} = \pi/6 - 0.523 = 0.11\%$$

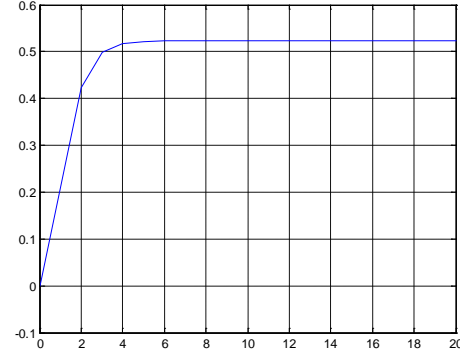


Figure 6: Response to quadratic torque

It is observed that when the torque is quadratic in relation to the speed dv/dt is very steep even though we assumed that speed (v) follows a gradual increase function. The graph looks similar to case 1A where the speed (v) is a step change and $W_r \sim T$.

Case 3: Lookup Table

Let us assume that V is unknown. However we do

$$\text{know that } W_R = f(V) \Rightarrow W_R = \frac{V_{\infty}}{R}$$

Since there is no physical and direct relation between v and θ , we used the torque to produce the desired angle of displacement of the speedometer needle.

Now torque is calculated using the following expression where the torque produced is quadratic in relation to the car's speed.

$$\text{Torque } n = k_1(w_R) + k_2(w_R^2) + k_3(w_R^3) + k_4(w_R^4)$$

$$\text{Where } k_1 = 8; k_2 = \frac{8}{5} * 8 = 0.2; k_3 = \frac{0.2}{40} = 5 * 10^{-3}; k_4 = \frac{8}{40} = 1.25 * 10^{-4}$$

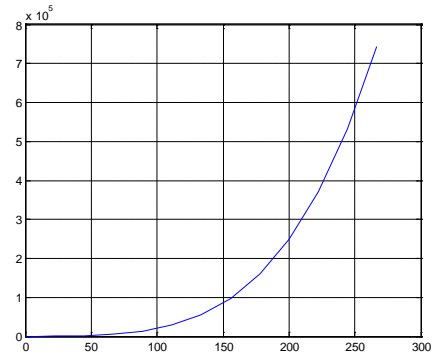


Figure 7: Speed Vs Torque plot

A MATLAB code was used to generate the values of the torque at given speeds. Figure 7 shows the plot between the Torque and the Speed. Using the values of Torque the values for the needle displacement θ can be calculated as shown in Table 1 below.

V_{∞}	W_R	Torque (T)	θ
0	0.00	0.00	0

10	22.22	361.90	$\pi/12$
□	□	□	□
120	266.67	43269.00	π

Conclusion:

In this work we have modeled a speedometer using LaPlace transforms for both linear and non-linear cases. We begin by developing the required transfer function and then testing its response under realistic linear and non-linear input scenarios. In all the cases discussed the results were very accurate as seen by the comparison between the expected and the obtained results and the steady state errors in all the cases was less than 0.15%.

References:

- [1] "How does a Speedometer work", Test & Measurement World, February 15, 2000.
- [2] "Modern Control Systems", Richard C. Dorf, Robert H. Bishop, Prentice Hall, 2008.

Biography

Navarun Gupta is an Assistant Professor in the Department of Electrical and Computer Engineering at the University of Bridgeport, Connecticut. He received a Ph.D. from Florida International University in 2003. His interests include the application of digital signal processing to acoustics and biosignals.

Sarosh Patel received the B.E. degree in Electrical and Electronics Engineering with Distinction from the Faculty of Engineering Osmania University, India in 2002, and M.S. degrees in Electrical Engineering and Technology Management from the School of Engineering, University of Bridgeport (UB), in 2006. He is currently pursuing Ph.D. in Computer Engineering at U.B.

Lawrence V. Hmurcik is Professor and chairman of Electrical Engineering at the University of Bridgeport, Bridgeport, CT. He earned his Ph.D. in semiconductor devices at clarkson University in 1980. He worked in diamond Shamrock's research division for 3 years before joining the University of Bridgeport in 1983. Dr. Hmurcik has 50 publications and 5 grants. He is also a professional consultant with 240 case entries, including 14 appearances in Court and Legal Depositions. Dr. Hmurcik's interests have changed over the years: starting in Solar Cell technology in 1977, Dr. Hmurcik is currently pursuing work in Medical Electronics and Electric Safety.