Contrasts between Maximum Power Transfer and Maximum Efficiency

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There is a trend in physics texts to present the concept of maximum power transfer without presenting an analysis of the maximum efficiency as well. Consider the circuit in Fig. 1. A battery (or generator) with internal resistance R_i supplies power to the load resistor R . The power transferred is V^2/R , and using voltage divider analysis

$$V = E \frac{R}{R + R_i} \quad (1)$$

so that

$$P = \frac{E^2 R}{(R + R_i)^2} \quad (2)$$

Using calculus, we can show that $R = R_i$ produces maximum power transfer. Also, several non-calculus methods can be applied to prove this point.¹ If we graph P vs. $\log(R/R_i)$, the maximum power point is clearly indicated at $R/R_i = 1$ (see Fig. 2). By using a logarithmic scale for resistance, P appears symmetric about the maximum power point.

This treatment is outlined in many physics textbooks. However, it overlooks the importance of power efficiency. If we define the efficiency η to be power output to our load relative to what our generator produces, then

$$\eta = \frac{P}{P + P_i} \times 100\% \quad (3)$$

and

$$\eta = \frac{R}{R + R_i} \times 100\% \quad (4)$$

When $R = R_i$, $\eta = 50\%$. Half of our power is wasted by the internal resistance of the generator, R_i . This is not the way to build a good generator.

To make this point clear to students, we cite the classic problem that Thomas Edison faced when he invented his light bulb. An excerpt by an author in *Scientific American*² will clarify this point: "Much has been written and said about Mr. Edison's lamps, but there has been wonderfully little produced in the way of a precise and reliable statement concerning the simple facts. Assuming that a Siemens or Brush machine is employed to generate the electric current, such a current would be

Fig. 1. A battery or DC generator of voltage E and internal resistance R_i is connected across load R . For the present analysis, an AC generator can also be used.

obtained with a loss of about 40 per cent of the mechanical energy applied to the driving pulley of the machine. To operate 12 lamps, therefore, we should have to apply one horse power to the pulley. To produce one horse power in a steam engine, about five pounds of coal must be burned. These five pounds of coal will yield 25 cubic feet of gas, which if burned in five gas burners will give 100 to 110 candles. The 12 Edison lamps will give 120 candles. This would show something in favor of the economy of the electric light, but when in fact everything in this regard is against the electric light, which demands more machinery, requires more skillful management, and presents an utter lack of storage capacity that secures efficiency and convenience for gas, we see that this trifling economy disappears."

First, we ask the students to support this author and prove that the efficiency of Edison's light bulb will be even less than the 60% claimed. By plotting P vs. R and also by using calculus, it is not difficult to obtain the value of 50% as the efficiency at *maximum power*. Then, we ask them to plot η vs. R for $R > R_i$. It soon becomes obvious that efficiencies greater than 50% are possible (see Fig. 2). Efficiency tends towards 100% as the load resistance becomes infinite. As Edison demonstrated, the most efficient generator is the one with the smallest internal resistance or largest load resistance.

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A generator cannot be 100% efficient. To do so, it would generate zero power. We can consider a simple generator used to power a house light. (Note: a generator is usually AC and not DC as we show in Fig. 1. We make this point clear to our students, but indicate that when using resistor circuits, there is no change in our basic analysis if we use RMS values.) Since house current has a typical RMS voltage of 115, we choose $E = 100$ to make the calculation simpler. With $E = 100$, the terminal voltage is given as

$$V = 100 \frac{R}{R + R_i} \quad (5)$$

Equations (4) and (5) indicate that terminal voltage and power efficiency are the same. This is only a coincidence (a result of our arbitrary choice of $E = 100V$), but it focuses attention on the fact that the percentage of the generator voltage applied across a load will determine the efficiency of the operation. If we consider a 100-W light bulb as a load with $R_i = 1$, then

$$100 = V^2/R \quad (6)$$

Solving (5) and (6), we find that V is 99 volts (99% efficient). This point is indicated in Fig. 2. [In Equation (6), R is the effective resistance of a light bulb. It relates current and voltage for only a single value of resistance at the steady state filament temperature. It does not account for the normally non-Ohmic behavior of a light bulb.^{3]}

Once students have grasped the concept of power transfer vs. efficiency, variations of the original problem are introduced. For example, the concept of "loading" can be introduced by holding R_i fixed and decreasing R . One variation which we stress pertains to transmission lines. A coaxial cable (such as those used for cable-TV) is a simple transmission line.⁴ It has an impedance to a TV signal of 75Ω . If we let $R_i = 75$, then our TV set must act as a 75 load resistor to obtain maximum power. But do we want maximum power or maximum efficiency? Since efficiency varies directly as the signal voltage, this could be a more important parameter. In practice, we want both a large signal strength (efficiency) and a large power transfer. If we define the relative power transfer as

$$\alpha = P/P_{max} \quad (7)$$

Fig. 2. The power transferred to a load resistor (P) and the efficiency (η) of this transfer are plotted vs. the relative load resistance R/R_i . The x-axis is logarithmic to show symmetry around the maximum power point $R/R_i = 1$. Points at $R/R_i = 100$ are also noted.

then we maximize both parameters when α and η are equal. Using our previous results, this gives

$$R = 3R_i \quad (8)$$

and

$$\eta = 75\% \quad (9)$$

In Fig. 2., this point corresponds to the intersection of our plots of efficiency and power.

Acknowledgement

We wish to thank Prof. M. Hettinger for help in the preparation of this note. □

References

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